Signal Analysis & Communication ECE 355

Ch. 3.4: Convergence of Fourier Series.

Lecture 16

16-10-2023

Ch. 3.4: CONVERGENCE OF FOURIER SERIES

Recall: Fourier Series Representation of a CT Periodic Sig.

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad -① \quad \text{(SYNTHESIS)}$$

where

$$a_k = \frac{1}{T} \int_T x_{(t)} e^{-jk\omega_0 t} dt \quad -② \quad \text{(ANALYSIS)}$$

$\underbrace{\text{(Almost All)}}$ periodic signals can be represented using Fourier Series (FS)

Q. Which periodic signals have a Fourier Series representation?

- To answer this, lets examine the problem of approximating a given periodic sig. $x_{(t)}$ by considering a finite number of harmonically related complex exponentials.

i.e.,

$$x_{N(t)} = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \simeq x_{(t)} \quad -③$$

- The error introduced due to this approximation is given as:

$$e_{N(t)} = x_{(t)} - x_{N(t)}$$

- The quantitative measure of the size of the approximation error is:

$$E_N = \int_T |e_{N(t)}|^2 dt$$

- Now as $N\uparrow$, $E_N\downarrow$ as more terms will be added to $x_{N(t)}$ in ③
- Every continuous periodic signal has a Fourier Series Representation for which $E_N \to 0$ as $N \to \infty$.
- This is also true for many periodic signals with discontinuities

Not all! (for some a_k may be infinite, i.e., diverge)

Q. Which are those periodic signals, which can't be represented via Fourier Series?

- There are two different classes of conditions that a periodic sig. must satisfy to guarantee that it can be represented by a FS.

① If $x_{(t)}$ has a finite energy over a single period.

$$\int_T |x_{(t)}|^2 dt < \infty$$

$$\Rightarrow a_k = \frac{1}{T}\int_T x_{(t)} e^{-jk\omega_0 t} dt \rightarrow \text{Finite}$$

- Again, if

$$x_{N(t)} = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- $E_N \rightarrow 0$, as $N \rightarrow \infty$

Remember:

- $E_N = 0$ doesnot imply that the sig. $x_{(t)}$ & its FS representation are equal at every value of 't'.
- It implies that there is no energy in their difference.
- Now since the physical systems respond to sig. energy — ∴ from this perspective $x_{(t)}$ & its FS representation are indistinguishable.
- Most of the periodic signals that we consider do have finite energy over a single period, ∴ they can be represented via FS.
- Example - previous lecture.

② If $x_{(t)}$ satisfies "Dirchlet Conditions".

(satisfied by essentially all periodic signals)

- Dirichlet Conditions

I. The CT periodic sig. is absolutely integrable over a period.

$$\int_T |x_{(t)}|\, dt < \infty \quad \text{— (4)}$$

- This guarantees that a_k will be finite.

$$a_k = \frac{1}{T}\int_T x_{(t)} e^{-jk\omega_0 t}\, dt$$

$$|a_k| = \frac{1}{T}\left| \int_T x_{(t)} e^{-jk\omega_0 t}\, dt \right|$$

$$\leq \frac{1}{T}\int_T |x_{(t)}| \underbrace{|e^{-jk\omega_0 t}|}_{=1}\, dt$$

$$\leq \frac{1}{T}\int_T |x_{(t)}|\, dt$$

$$\leq \infty$$

- A periodic sig. that violates condition given in (4):

$$x_{(t)} = \frac{1}{t} \quad , \quad 0 < t \leq 1$$

II. In any finite interval of time, $x_{(t)}$ is of bounded variation.

- That is, there is no more than a finite number of maxima & minima during any single period of sig.

III. In any finite interval of time, there are only a finite number of discontinuities.

- Moreover each of these discontinuities is finite.

CONCLUSIONS

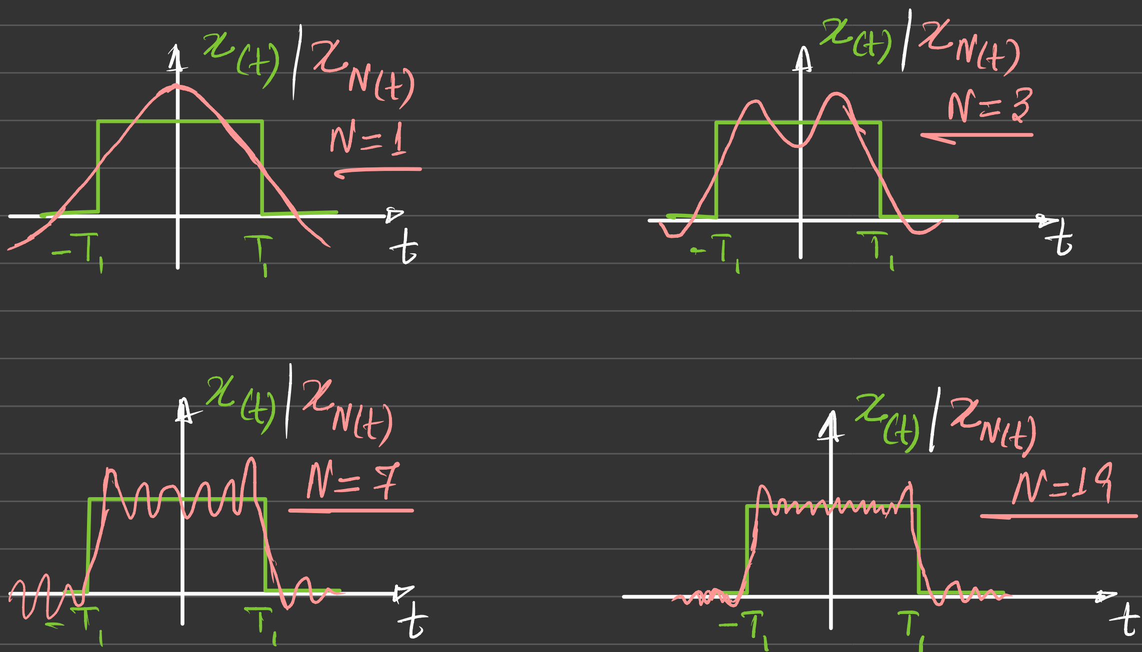
① For a continuous periodic sig. (i.e., which doesnot have discontinuities, the FS representation converges & equals the original sig. at every value of 't'.

② For a periodic sig. with finite number of discontinuities in each period, the FS representation equals the sig. everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the sig. on either side of the discontinuity.

(In this case, the difference b/w the original sig. & its FS representation contains no energy, & so, the two signals can be thought of as being 'the same' for all practical purposes.)

SOME ADDITIONAL UNDERSTANDING VIA GIBBS PHENOMENON

- Truncated FS approximation.



- As N↑, the ripples in the partial sum becomes compressed towards the discontinuity.
- For any finite value of N, the peak amplitude of the ripples remains constant (by about 9% of the height of the discontinuity).

- The truncated FS approximation $x_{N(t)}$ of a discontinuous sig. $x_{(t)}$ will in general exhibit high freq. ripples & overshoot $x_{(t)}$ near the discontinuities.

- For such an approximation, a large enough value of N should be chosen in practice so as to guarantee that the total energy in these ripples is insignificant.